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THE DISTANCE BETWEEN CERTAIN n-DIMENSIONAL BANACH SPACES[†]

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ABSTRACT

If E and F are n-dimensional Banach spaces, if E has cotype 2, and if the ball of F^* has a small number of extreme points, then the Banach–Mazur distance $d(E,F) \leqq C\sqrt{n}\log n$. The techniques lead to the formally stronger result: If E and F^* have type 2 constants a and b, respectively, then $d(E,F) \leqq \sqrt{n}(a+b)$. If E is n-dimensional, the identity map on E, when restricted to a large subspace of E, factors through l_∞^n with norm $C\sqrt{n}$.

§0. Introduction

Given Banach spaces E and F, the Banach–Mazur distance between them is defined by $d(E,F) = \inf\{\|T\|\,\|T^{-1}\| \mid T: E \to F\}$. Geometrically, this says that, for any $\varepsilon > 0$, there is a map $S: E \to F$ so that $B_F \subset S(B_E) \subset (d(E,F)+\varepsilon)B_F$, where B_X denotes the unit ball of the space X. If E is an n-dimensional space, the F. John lemma [12] implies that $d(E,l_2^n) \leqq \sqrt{n}$. Thus, for any pair of n-dimensional spaces, $d(E,F) \leqq d(E,l_2^n)d(l_2^n,F) \leqq n$. In the other direction, it is known that $d(l_1^n,l_2^n) = \sqrt{n}$. Therefore, if $\mathscr{F}_n$ denotes the collection of all n-dimensional Banach spaces endowed with the Banach–Mazur distance, we have $\sqrt{n} \leqq \operatorname{diam}\mathscr{F}_n \leqq n$.

One case is completely settled. Motivated by work of Asplund [1], W. Stromquist [24] has recently proved that $\operatorname{diam}\mathscr{F}_2 = 3/2$. In fact, he was able to show that the Banach–Mazur radius of $\mathscr{F}_2$ is $\sqrt{3/2}$.

Starting with two papers of Gurarii, Kadec and Macaev, distances within specific classes of spaces have been examined. In [10] the correct asymptotic distance between l_p^n and l_q^n were computed. In [11], an estimate was given for

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$d(l_1^n, E)$ in the case that E has a symmetric basis. Recall that a basis $(e_i)_{i=1}^n$ is *symmetric* if, for any sequence of scalars, (a_i), and any permutation, σ, of $\{1, \cdots, n\}$, we have $\|\Sigma a_ie_i\| = \|\Sigma |a_i| e_{\sigma(i)}\|$. This direction has been pursued more recently in the papers [4] and [28]. What is now known is that there is a universal constant, C (on the order of 10), such that $d(E, F) \leqq C\sqrt{n}(\log n)^2$ if E and F are n-dimensional symmetric spaces. We must point out that the presence of the logarithmic factor in this estimate is not known to be necessary. Such factors occur again in our estimates, and may simply be by-products of the proofs, rather than consequences of reality.

This sort of investigation has continued in several other special cases. For example, if E has a 1-unconditional basis, then the third-named author has recently shown that $d(l_p^n, E) \leqq C\sqrt{n(1+\log n)}$. This result, due to technical difficulties, has not yet been extended to the general case of $d(E, F)$ when E and F both have 1-unconditional bases. (A basis (e_i) is 1-*unconditional* if $\|\Sigma a_ie_i\| = \|\Sigma |a_i| e_i\|$ for all choices of scalars, (a_i).)

All of the results mentioned above are proved very constructively, and use quite specific orthogonal matrices. Somewhat more existential is the basic argument in [26]. There it is shown that, asymptotically, $d(C_p^n, C_q^n) \sim d(l_p^n, l_q^n)$. Here, C_p^n denotes the Schatten class of operators on l_2^n. That is, if $T: l_2^n \rightarrow l_2^n$, its norm in C_p^n is just $(\operatorname{tr}(T^*T)^{p/2})^{1/p}$. Notice that $\dim C_p^n$ is n^2, so that the estimates for $d(C_1^n, C_\infty^n) \sim (\dim C_1^n)^{1/4}$, for example.

Before we go on, we need some terminology which we shall employ throughout this work. Let X be a Banach space, and let $(r_i(t))$ denote the Rademacher functions on $[0, 1]$. X is said to have type 2 with constant $K^{(2)}(X)$ (respectively cotype 2 with constant $K_{(2)}(X)$) if

$$\left(\int_0^1 \left\|\sum r_i(t)x_i\right\|^2 dt\right)^{1/2} \leqq K^{(2)}(X)\left(\sum \|x_i\|^2\right)^{1/2},$$

$$\left(\text{respectively,} \quad \geqq (1/K_{(2)}(X))\left(\sum \|x_i\|^2\right)^{1/2}\right).$$

The constants $K^{(2)}$ and $K_{(2)}$ are assumed to be the smallest which work. Since, for $1 \leqq p < \infty$, $(\int \|\Sigma r_i(t)x_i\|^p dt)^{1/p} \sim (\int \|\Sigma r_i(t)x_i\|^2 dt)^{1/2}$, [13], we need not worry too much about the norm used on the left-hand side.

We shall, of course, also need some facts from the theory of operator ideals. Most of what we use can be found in [16] or [21] as well as in many of the other references cited here. Let $(x_i)_{i=1}^m \subset X$. Define

$$\varepsilon_p(x_i) = \sup\{(\Sigma |x^*(x_i)|^p)^{1/p} \mid \|x^*\| = 1, x^* \in X^*\}.$$

An operator, $u : X \to Y$, is p-summing if there is a (smallest) constant, $\pi_p(u)$, such that for all choices of $(x_i) \subset X$, we have

$$\left(\sum \|u(x_i)\|^p\right)^{1/p} \leqq \pi_p(u)\varepsilon_p(x_i).$$

The constant, $\pi_p(u)$, defines a norm, and we always have $\pi_p(vuw) \leqq \|v\|\,\|w\|\,\pi_p(u)$. In case X and Y are finite dimensional, the space of 1-summing operators from X to Y has, as its dual, the space of ∞-factorizable operators. That is, defining $\gamma_\infty(u) = \inf\|\beta\|\,\|\alpha\|$ such that $\alpha : X \to l_\infty$, $\beta : l_\infty \to Y$ and $u = \beta \circ \alpha$, we have $\gamma_\infty(u) = \sup\{|\operatorname{tr} uv| \mid \pi_1(v) = 1\}$ and $\pi_1(u) = \sup\{|\operatorname{tr} uv| \mid \gamma_\infty(v) = 1\}$. Another norm we shall need is $\gamma_2(u)$. Analogously, $\gamma_2(u) = \inf\|\beta\|\,\|\alpha\|$ such that $\alpha : X \to l_2$, $\beta : l_2 \to Y$ and $\beta \circ \alpha = u$. In case u is the identity on X, $\gamma_\infty(u)$ is the projection constant of X and $\gamma_2(u)$ is the distance from X to $l_2^{\dim X}$. Finally, we need the nuclear norm. Let $u : X \to Y$, where the spaces are finite dimensional. Then

$$\nu(u)(= i_1(u)) = \inf\left\{\sum \lambda_i \;\middle|\; u(x) = \sum \lambda_i x_i^*(x) y_i,\ \lambda_i \geqq 0, \|x_i^*\| = \|y_i\| = 1\right\}.$$

This norm is the dual to the operator norm: $\nu(u) = \sup\{|\operatorname{tr} uv| \mid \|v\| = 1\}$.

In §1, we abandon the symmetry and lattice assumptions above and prove a somewhat different sort of distance estimate: If the unit ball of E and unit ball of F^* both have a small number of extreme points, say less than n^a for some a, then the distance from E to F is bounded by $C\sqrt{n(1+\log n)}$, where C depends only on a above. This result has its roots in [26], where distance estimates are obtained by "unitary factorizations" of operators through Euclidean spaces. In [3], Chevet proved an inequality involving "Gaussian factorizations" which allows us to extend this result. The idea of using the Chevet theorem in this sort of factorization question originated in an early version of the paper by Benyamini and Gordon [2]. We appreciate their letting us have a preprint of that paper. We are also grateful to G. Pisier for showing us an inequality in Marcus and Pisier [18] which relates Gaussian and unitary averages, and hence allows the application of the factorization scheme to our problem. The extension which occurs is: if E and F are n-dimensional, then $d(E,F) \leqq \sqrt{n}(K^{(2)}(E) + K^{(2)}(F^*))C$, where C is a universal constant. In case the spaces E and F have enough symmetries, this result may be improved by replacing $\sqrt{n}$ by $\max(d(E, l_2^n), d(F, l_2^n))$.

The second section contains various results related to this work. For example, if E is an arbitrary n-dimensional space, it has an $n/10$-dimensional subspace whose identity factors through l_∞^n as $u \circ v$ with $\|u\|\|v\| \leqq C\sqrt{n}$ (C is universal). We conclude the second section with estimates of the Levy mean of the norm of an n-dimensional space with respect to the F. John ellipsoid.

§1. Distance questions related to type 2 constants

The main result of this section is Theorem 1, which says that if E and F are n-dimensional, then $d(E,F) \leqq C\sqrt{n}(K^{(2)}(E^*) + K^{(2)}(F))$. The motivation for this result lies in the following proposition, which is a modification of the main result of [26].

Theorem 1 below is a strengthening of Proposition 1. If B_{F^*} has a small number, say $2m$, of extreme points, then F embeds isometrically into l_∞^m. Since l_∞^m has type 2 constant on the order of $\sqrt{\log m}$, so does F. In particular, if $m \leqq n^a$, then F has type 2 with constant smaller than $\sqrt{a \log n}$.

PROPOSITION 1. *Let* $\dim E = \dim F = n$, *suppose that* $|\operatorname{Ext} B_E| \leqq n^\alpha$ *and* $|\operatorname{Ext} B_{F^*}| \leqq n^\beta$. *Then there is a constant* $C = C(\alpha, \beta)$, *not depending on* E, F *or* n, *such that* $d(E,F) \leqq C(\alpha,\beta)\sqrt{n(1+\log n)}$.

PROOF. Let $\mathscr{E}$ and $\tilde{\mathscr{E}}$ denote ellipsoids such that $(1/\sqrt{n})\mathscr{E} \subset B_E \subset \mathscr{E}$ and $\tilde{\mathscr{E}} \subset B_F \subset \sqrt{n}\tilde{\mathscr{E}}$. For example, by F. John's lemma, $\mathscr{E}$ may be taken to be the minimal volume ellipsoid containing B_E and $\tilde{\mathscr{E}}$ the maximal volume ellipsoid contained in B_F, [12]. After an affine transformation, we may as well assume that $\mathscr{E} = \tilde{\mathscr{E}}$, so that we have $(1/\sqrt{n})\mathscr{E} \subset B_E \subset \mathscr{E} \subset B_F \subset \sqrt{n}\mathscr{E}$. If we let $|\cdot|_2$ denote the Euclidean norm on $\mathbf{R}^n$ given by this ellipsoid, and denote the resulting Euclidean space by H, we consider only maps from E to F of the form $E \xrightarrow{i} H \xrightarrow{u} H \xrightarrow{j} F$. Here i and j denote the formal identity maps and u, to be chosen, is a unitary operator on H. We have, from above, $\|i\| = \|j\| = 1$ and both $\|j^{-1}\|, \|i^{-1}\| \leqq \sqrt{n}$. Of course, $\|u\| = 1 = \|u^*\|$. Thus, if we set $A^{-1} = i^{-1} \circ u^* \circ j^{-1}$, we have $\|A^{-1}\| \leqq n$, independent of the choice of u. To estimate the norm of $A = j \circ u \circ i$, we need some notation. Let U denote the group of unitary operators on H, and let μ denote the normalized Haar measure on U. Just as in [26], for any constant a and vectors $x, y \in H$ having norm 1, a direct calculation shows that

$$\mu\{u \in U \mid |\langle ux, y\rangle| \geqq a\} \leqq Ce^{-na^2/2},$$

where C is a universal constant (sufficient is $C = 4$). This sort of estimate is convenient for computing $\|A\|$, since $\|A\| = \max\{|\langle Ax, y'\rangle| \mid x \in \operatorname{Ext} B_E, y' \in \operatorname{Ext} B_{F^*}\}$. For $x \in \operatorname{Ext} B_E$ and $y' \in \operatorname{Ext} B_{F^*}$, then, we have

$$\mu\{u \in U \mid |\langle ux, y'\rangle| \geqq a\,|x|_2|y'|_2\} \leqq Ce^{-na^2/2}.$$

For $\|x\|_E = 1$ and $\|y'\|_{F^*} = 1$, we have $1 \geqq |x|_2 \geqq 1/\sqrt{n}$ and $1 \geqq |y'|_2 \geqq 1/\sqrt{n}$, respectively. In particular, this gives us $\mu\{u \mid |\langle ux, y'\rangle| \geqq a\} < Ce^{-na^2}$ if $x \in \operatorname{Ext} B_E$ and $y' \in \operatorname{Ext} B_F$. Therefore, we have that

$$\mu\{u \mid \|A\| < a\} \geqq 1 - C|\operatorname{Ext} B_E|\,|\operatorname{Ext} B_{F^*}|\,e^{-na^2}.$$

For the right-hand side to be positive we simply need $|\operatorname{Ext} B_E|\,|\operatorname{Ext} B_{F^*}| < (1/C)e^{na^2}$. Taking the log of both sides, and recalling the hypothesis, we see that we need $\alpha \log n + \beta \log n < na^2 - \log C$, so that, in particular for $n \geqq 3$, it is sufficient to have

$$a = \left(\frac{(\alpha + \beta + \log C)\log n}{n}\right)^{1/2}.$$

What we have shown is that there is a unitary operator u so that

$$\|A\| = \|j \circ u \circ i\| < \sqrt{\alpha + \beta + \log C}\sqrt{\frac{\log n}{n}}$$

and so we have $d(E, F) \leqq \|A\|\,\|A^{-1}\| \leqq \sqrt{\alpha + \beta + \log C}\sqrt{n \log n}$. This completes the proof with $C(\alpha, \beta) = \sqrt{\alpha + \beta + \log C}$.

We now proceed to the main result of this section. We need a definition and some results from [7]. Let $T : l_2^n \to E$, and define

$$M_T = \left(\int_{S^{n-1}} \|Tx\|^2 m(dx)\right)^{1/2}, \qquad l(T) = \sqrt{n}M_T.$$

If γ_n denotes normalized Gaussian measure on l_2^n, then $l(T) = (\int_{l_2^n} \|Tx\|^2 d\gamma_n)^{1/2}$. If (e_i) denotes the natural basis for l_2^n, and if (g_i) is an independent sequence of Gaussian random variables with mean zero and variance 1, then we also have $l(T) = \int_\Omega \|\Sigma\, g_i(w)Te_i\|^2 dw)^{1/2}$. From this last statement, since Gaussian random variables can replace Rademacher functions in the definition of type 2 and cotype 2 constants, we see that $l(T) \leqq K^{(2)}(E)(\Sigma\|Te_i\|^2)^{1/2} \leqq K^{(2)}(E)\pi_2(T)$.

LEMMA 1. *Let $T : l_2^n \to E$. Then, $l(T) \leqq K^{(2)}(E)\pi_2(T^*)$.*

PROOF. Let $E^* \xrightarrow{v} l_\infty^m \xrightarrow{\Delta} l_2^m \xrightarrow{w} l_2^n$ be a good 2-nuclear factorization of T^* (e.g., [21]). That is, $\|v\| = \|w\| = 1$ and $\|\Delta\| < (1 + \varepsilon)\pi_2(T^*)$, where $\Delta(e_i) = \delta_i e_i$. Thus $\|\Delta\| = (\Sigma|\delta_i|^2)^{1/2}$. Then $(\int\|Tx\|^2 d\gamma_n)^{1/2} = (\int\|v^*\Delta^* w^* x\|^2 d\gamma_n)^{1/2}$ by [7]. Since $\|w^*\| = 1$, we have

$$\left(\int \| Tx \|^2 d\gamma_n\right)^{1/2} \leqq \left(\int \| v^*\Delta^* w^* x \|^2 d\gamma_n\right)^{1/2}$$

$$= \left(\int_\Omega \left\| \sum_{i=1}^m g_i(w) v^*\Delta^* e_i \right\|^2 dw\right)^{1/2}$$

$$\leqq K^{(2)}(E)\left(\sum_{i=1}^m \| v^*\Delta^* e_i \|^2\right)^{1/2} \leqq K^{(2)}(E)\left(\sum_{i=1}^m \| \Delta^* e_i \|^2\right)^{1/2}$$

$$= K^{(2)}(E)\|\Delta^*\| < K^{(2)}(E)(1+\varepsilon)\pi_2(T^*).$$

Since ε is arbitrary, we have the result desired.

Last, we need a lemma well known to many people.

LEMMA 2. *Let $u: l_2^n \to E$ such that $u(B_{l_2^n})$ is the ellipsoid of maximal volume in B_E. Then $\pi_2(u) = \pi_2(u^{-1}) = \sqrt{n}$.*

PROOF. By definition, $\| u \| \leqq 1$, so $\pi_2(u) \leqq \| u \| \pi_2(l_2^n) = \sqrt{n}$. As in, for example, [9], [16], there are positive scalars λ_i with $\Sigma \lambda_i = n$ and points $\{x_i\}_{i=1}^m \subset E$ so that $\| x_i \|_E = 1 = \| x_i \|_{E^*} = \| u^{-1}(x_i) \|$. Here the pairing is given by $x_i(x) = \langle u^* x_i, u^{-1}(x)\rangle$. Further, for $x \in E$, $x = \Sigma \lambda_i x_i(x) x_i$. Let $\{z_i\}_{i=1}^l \subset E$ so that $(\Sigma |\langle x', z_i\rangle|^2)^{1/2} \leqq \| x' \|_{E^*}$ for each $x' \in E^*$. We have

$$\sum_{j=1}^l \| u^{-1}(z_j) \|^2 = \sum_{j=1}^l \langle u^{-1}(z_j), u^{-1}(z_j)\rangle = \sum_{i=1}^m \lambda_i \sum_{j=1}^l | x_i(z_j) |^2 \leqq n,$$

by the choice of $\{z_j\}$. Thus, by definition, we have $\pi_2(u^{-1}) \leqq \sqrt{n}$. Since $\operatorname{tr}(u \circ u^{-1}) = n \leqq \pi_2(u)\pi_2(u^{-1})$, we see that $\pi_2(u) = \pi_2(u^{-1}) = \sqrt{n}$ as desired.

THEOREM 1. *Let $\dim(E) = \dim(F) = n$. Then $d(E,F) \leqq c\sqrt{n}(K^{(2)}(E^*) + K^{(2)}(F))$, where c is a universal constant.*

PROOF. Let $v: l_2^n \to E$ and $w: l_2^n \to F^*$ have the property that $v(B_{l_2^n})$ is the maximal volume ellipsoid in B_E and $w(B_{l_2^n})$ is the maximal volume ellipsoid in B_{F^*}. We will select a unitary map, u, on l_2^n and set $T = (w^{-1})^* \circ u \circ (v^{-1})$. Then $T^{-1} = v \circ u^* \circ w^*$ has norm $\leqq 1$. First we need a concrete representation of T. Let $x_i' = (v^{-1})^*(e_i)$ and $y_i = (w^{-1})^*(e_i)$ for $i = 1, 2, \cdots, n$. Then, if $u = (u_{ij})$, we have $T = T_u = \Sigma_{i,j=1}^n u_{ij} x_i' \otimes y_j$. If μ denotes normalized Haar measure on the unitary group, $\mathcal{U}_n$, a recent result of Marcus and Pisier [18, V.2.4] says that

$$\left(\int_{\mathcal{U}_n} \| T_u \|^p \mu(du)\right)^{1/p} \leqq \frac{c_p}{\sqrt{n}}\left(\int \left\| \sum g_{ij}(w) x_i' \otimes y_j \right\| dw\right),$$

where c_p depends only on p and $\{g_{ij}\}$ are independent, $N(0,1)$ Gaussian random variables. Using the inequality due to Chevet, [3], we see that

$$\left(\int_{\mathcal{U}_n} \| T_u \|^p \mu(du)\right)^{1/p} \leqq$$

$$\leqq \frac{c_p}{\sqrt{n}} \left(\varepsilon_2(x_i') \left(\int \left\| \sum g_i y_i \right\|^2 dw \right)^{1/2} + \varepsilon_2(y_i) \left(\int \left\| \sum g_i x_i' \right\|^2 dw \right)^{1/2} \right).$$

Recall that $\varepsilon_2(x_i') = \sup_{\|x\|=1} (\Sigma |\langle x_i', x\rangle|^2)^{1/2}$, so we get $\varepsilon_2(x_i') \leqq \| v^{-1} \|$. Similarly, $\varepsilon_2(y_i) \leqq \| w^{-1} \|$. As above, $(\int \| \Sigma g_i y_i \|^2 dw)^{1/2} = l((w^{-1})^*)$ and $(\int \| \Sigma g_i x_i' \|^2 dw)^{1/2} = l((v^{-1})^*)$. So we have

$$(1) \qquad \left(\int_{\mathcal{U}_n} \| T_u \|^p\right)^{1/p} \leqq \frac{c_p}{\sqrt{n}} \{ \| v^{-1} \| l((w^{-1})^*) + \| w^{-1} \| l((v^{-1})^*) \}.$$

By Lemma 1, we have $l((w^{-1})^*) \leqq K^{(2)}(F) \pi_2(w^{-1})$, and $l((v^{-1})^*) \leqq K^{(2)}(E^*) \pi_2(v^{-1})$. Using Lemma 2, we get $\int_{\mathcal{U}_n} \| T_n \| \mu(du) \leqq C\{ \| v^{-1} \| K^{(2)}(F) + \| w^{-1} \| K^{(2)}(E^*) \}$. Since $\| v^{-1} \| \leqq \pi_2(v^{-1})$, etc., we finally get $\min \| T_u \| \leqq \int_{\mathcal{U}_n} \| T_n \| d\mu \leqq c\sqrt{n}\{K^{(2)}(E^*) + K^{(2)}(F)\}$, as desired.

In the proof above, notice that, if the maximal ellipsoid yields $\| v^{-1} \| = d(E, l_2^n)$, and $\| w^{-1} \| = d(F, l_2^n)$, then we get the estimate

$$d(E,F) \leqq C(K^{(2)}(E^*) + K^{(2)}(F)) \max\{ d(X, l_2^n), d(Y, l_2^n) \}.$$

This situation occurs if E and F have enough symmetries (see, e.g. [9]).

Corollary 1. *If* $\dim E = \dim F = n$, *and if* E *and* F *have enough symmetries, then*

$$d(E,F) \leqq C(K^{(2)}(E^*) + K^{(2)}(F)) \max\{ d(E, l_2^n), d(F, l_2^n) \}.$$

This corollary is, in a sense, best possible, due to the following.

Proposition 2. *Let* $\dim(E) = \dim(F) = n$. *Then*

$$d(E,F) \geqq \tfrac{2}{5}(K^{(2)}(E^*) K^{(2)}(F))^{-3/2} \max\{ d(E, l_2^n), d(F, l_2^n) \}.$$

Proof. It is true that $K_{(2)}(E) \leqq K^{(2)}(E^*)$ and $K_{(2)}(F^*) \leqq K^{(2)}(F)$. By a result of Pisier, [23], if $u : F \to E$, then $\gamma_2(u) \leqq \frac{5}{2}(K_{(2)}(F^*) K_{(2)}(E))^{3/2} \| u \|$. Let u be chosen so that $\| u \| \| u^{-1} \| = d(E,F)$. By the Pisier result, $u = \beta \circ \alpha$ where $\alpha : F \to l_2^n$, $\beta : l_2^n \to E$, and $\| \beta \| \| \alpha \| \leqq (\frac{5}{2})(K_{(2)}(F^*) K_{(2)}(E))^{3/2} \| u \|$. We have $d(l_2^n, E) \leqq \| \beta \| \| \alpha \circ u^{-1} \|$. That is, $\max\{ d(E, l_2^n), d(F, l_2^n) \} \leqq \| \alpha \| \| \beta \| \| u^{-1} \|$. The proof is complete.

It may be that the result of Corollary 1 holds in general. That is,

PROBLEM 1. Is there a constant C so that, if $\dim E = \dim F = n$, then $d(E,F) \leqq c(K^{(2)}(E^*) + K^{(2)}(F)) \max\{d(E, l_2^n), d(F, l_2^n)\}$?

COROLLARY 2. *Let $d_X = d(l_2^n, X)$ and let $k(X)$ be the largest dimension of a subspace of a Banach space X which is 2-isomorphic to the euclidean space l_2^k. Let E and F be n-dimensional Banach spaces. Then*

$$d(E,F) \leqq C\frac{d_E \cdot d_F}{n}\{(\sqrt{k(F)} + \sqrt{k(E^*)})(\sqrt{k(E)} + \sqrt{k(F^*)})\}$$

where C is some universal constant.

PROOF. We have to return to the proof of Theorem 1. First, we use new embeddings $v : l_2^n \to E$ and $w : l_2^n \to F^*$ which give the distances d_E and d_F. More precisely, let $1/d_E \|x\| \leqq \|vx\| \leqq \|x\|$ and $1/d_F \|x\| \leqq \|wx\| \leqq \|x\|$. Then, by [6], §2, there exists an absolute constant c such that $k(E) \geqq cnM_v^2$ and $k(F^*) \geqq cnM_w^2$ (we recall that as before $M_v^2 = \int_{S^{n-1}} \|vx\|^2 m(dx)$). Similarly

$$k(E^*) \geqq cn\left(\frac{M_{(v^*)^{-1}}}{d_E}\right)^2 \quad \text{and} \quad k(F) \geqq cn\left(\frac{M_{(w^*)^{-1}}}{d_F}\right)^2.$$

We use all these inequalities to continue inequality (1) (we use the same letter c for different universal constants),

$$\begin{aligned}\left(\int_{\mathcal{U}_n} \|Tu\|^2 \mu(du)\right)^{1/2} &\leqq C\{\|v^{-1}\| M_{(w^{-1})^*} + \|w^{-1}\| M_{(v^{-1})^*}\} \\ &\leqq C\left\{ d_E d_F \sqrt{\frac{k(F)}{n}} + d_F d_E \sqrt{\frac{k(E^*)}{n}}\right\}.\end{aligned} \tag{2}$$

Now we apply the same reasoning to the operators $\|T_u^{-1}\|$. Instead of (1) we will obtain

$$\left(\int_{\mathcal{U}_n} \|T_u^{-1}\|^2 \mu(du)\right)^{1/2} \leqq \frac{c}{\sqrt{n}}\{\|w\| l(v) + \|v\| l(w)\} \leqq c\{M_v + M_w\}$$

$$\leqq c\left\{\sqrt{\frac{k(E)}{n}} + \sqrt{\frac{k(F^*)}{n}}\right\}.$$

So

$$d(E;F) \leqq \min \| T_u \| \cdot \| T_u^{-1} \| \leqq \int_{U_n} \| T_u \| \cdot \| T_u^{-1} \| d\mu(u)$$

$$\leqq \left(\int_{U_n} \| T_u \|^2 d\mu \right)^{1/2} \left(\int_{U_n} \| T_u^{-1} \|^2 d\mu \right)^{1/2}$$

and we have proved the corollary.

If we apply Theorem 1 to the estimation of $d(E, l_1^n)$, we simply get $d(E, l_1^n) \leqq C\sqrt{n}(\sqrt{\log n} + K^{(2)}(E))$.

This estimate can be improved as follows:

THEOREM 2. *For every $\alpha > 1$, there is a function $\varphi = \varphi_\alpha$, such that, for every m, $\varphi(n)/\log^{(m)}(n) \to 0$ as $n \to \infty$, such that $K^{(2)}(E) \leqq \alpha$ and $\dim E = n$ implies that $d(E, l_1^n) \leqq \varphi(n)\sqrt{n}$. Here $\log^{(m)}(n) = \log(\log(\cdots(\log n))\ldots)$ (m-iterates).*

Before we prove the theorem, we define φ. Let $\log x$ denote $\log_a x$ where $a = (e/2)^{1/2}$, so that $a^t \leqq e^{t/2}/4$ for $t \geqq 4$.

LEMMA 3. *Let $\alpha > 1$. There exist $q_0 \geqq 2$ and $\beta \geqq 1$ such that, for every finite dimensional space X with $K^{(2)}(X) \leqq \alpha$, we have $K_{(q_0)}(X) \leqq \beta$.*

PROOF. If not, let $\dim X_n < \infty$ and $K^{(2)}(X_n) \leqq \alpha$ for every n so that $K_{(n)}(X_n) > n$. Then $K^{(2)}((\Sigma X_n)_2) \leqq \alpha$, so $K_{(q)}((\Sigma X_n)_2) < \infty$, since an infinite-dimensional space with type $p > 1$ has cotype $q < \infty$ (e.g. [21]). This is a contradiction.

Now let $\mathscr{E}$ denote the maximal ellipsoid contained in B_{E^*}, and let $u^{*-1} : l_2^n \to E^*$ be the operator defining $\mathscr{E}$. Then $u : l_2^n \to E$ is defined in the natural way. We have as before, of course, that $\|(u^{-1})^*\| = 1$ and $\pi_2((u^{-1})^*) = \pi_2(u^*) = \sqrt{n}$.

LEMMA 4. *With the notation above, $\pi_{q_0,2}(u) \leqq \alpha\beta\sqrt{n}$.*

PROOF. By proposition 5 and corollary 6 of [15], we have $\pi_{q_0,2}(u) \leqq K_{(q_0)}(E) l(u) \leqq K_{(q_0)}(E) K^{(2)}(E) \pi_2(u^*)$. That is, $\pi_{q_0,2}(u) \leqq \alpha\beta\sqrt{n}$.

Now we define φ. Let t_0 be chosen so that $t \geqq t_0$ implies that $(\log t^{\alpha^2\beta^2})^{q_0/2} < t$. Let $\Psi(x) = (\log x^{\alpha^2\beta^2})^{q_0/2}$ and define

$$\tilde{\varphi}(n) = \min\{k : \underbrace{\Psi \circ \cdots \circ \Psi}_{k \text{ times}}(n)\} \leqq t_0^{\alpha^2\beta^2}.$$

Finally, $\varphi(n) = \sqrt{\tilde{\varphi}(n)}\,\alpha\beta\sqrt{\log t_0} + \alpha)$.

If F is a k-dimensional subspace of l_2^n, and if $G = u(F) \subset E$, M_G will denote the mean value of $\|u(x)\|$ on the sphere, $S(F) \subset l_2^n$. That is

$$M_G = \left(\int_{S^{k-1}} \|u(x)\|^2 m(dx)\right)^{1/2} = \frac{1}{\sqrt{k}}\left(\int_{l_2^k} \|u(x)\|^2 d\gamma_k\right)^{1/2}$$

By results above $M_G \leqq \sqrt{(n/k)}\, K^{(2)}(E)$.

PROOF OF THEOREM 2. We first define an orthonormal basis (f_n) for l_2^n. Let f_n be chosen so that

$$\|uf_n\| = \max\{\|ux\| : \|x\| = 1\}.$$

Having chosen $f_n, \cdots, f_{n-j+1}$, let f_{n-j} satisfy

$$\|uf_{n-j}\| = \max\{\|ux\| : \|x\| = 1, \langle x, f_i\rangle = 0 \text{ for } i > n-j\}.$$

What we have is $\|uf_j\| = \|u \mid [f_n, \cdots, f_{j+1}]^\perp\|$ for each j. Set $m_1 = n$, and let $J_1 = \{j : \|uf_j\| \leqq (n/\log m_1)^{1/2}\}$, and let $E_1 = \operatorname{span}(f_j)_{j\in J_1}$. We clearly have $\|u \mid E_1\| \leqq (n/\log m_1)^{1/2}$. Proceed inductively: suppose we have $k \geqq 1$, sets $J_1, J_2, \cdots, J_k$, subspaces $E_1, \cdots, E_k$, and integers $m_1 > m_2 > \cdots > m_k$ defined. Let $m_{k+1} = \dim((E_1 \oplus \cdots \oplus E_k)^\perp)$, and let

$$J_{k+1} = \{j : (n/\log m_k)^{1/2} < \|uf_j\| \leqq (n/\log m_{k+1})^{1/2}\}.$$

Also let $E_{k+1} = \operatorname{span}(f_j)_{j\in J_{k+1}}$. For $f_j \in (E_1 \oplus \cdots \oplus E_k)^\perp$, we must have $\|uf_j\| > (n/\log m_k)^{1/2}$, Lemma 4 gives us that $\alpha\beta\sqrt{n} \geqq \pi_{q_0,2}(u) \geqq m_{k+1}^{1/q_0}(n/\log m_k)^{1/2}$. Thus, $m_{k+1} \leqq (\alpha^2\beta^2 \log m_k)^{q_0/2}$. This procedure continues, then, as long as $m_{k+1} < m_k$. That is, it terminates after $k_0 \leqq \tilde{\varphi}(n)$ steps. From the definitions, we have $\|u \mid E_k\| \leqq (n/\log m_k)^{1/2}$ for $1 \leqq k \leqq k_0 - 1$, and $\|u \mid E_{k_0}\| \leqq \sqrt{n}$. Notice that $\dim E_{k_0} \leqq t_0^{\alpha^2\beta^2}$. We apply proposition 2.3 of [6] to E_k and the function $\|u(\cdot)\|$. This says that there is a unitary operator, U_k, on E_k so that $\|uU_kf_j\| \leqq M_{G_k} + \|u \mid E_k\|\varepsilon_k$ if $j \in J_k$ and $\dim E_k (\leqq m_k) \leqq a^{(\dim E_k)\cdot\varepsilon_k^2}$. We set $\varepsilon_k = (\log m_k/\dim E_k)^{1/2}$. Thus, it follows that $\|uU_kf_j\| \leqq (n/\dim E_k)^{1/2}[K^{(2)}(E) + 1]$, for $j \in J_k$. For $k = k_0$, we have

$$\begin{aligned}\|uU_{k_0}f_j\| &\leqq M_{G_{k_0}} + \|u\|\varepsilon_{k_0}\\ &\leqq \left(\frac{n}{m_{k_0}}\right)^{1/2}(K^{(2)}(E) + \sqrt{\log m_{k_0}})\\ &\leqq (n/m_{k_0})^{1/2}(K^{(2)}(E) + \alpha\beta\sqrt{\log t_0})\end{aligned}$$

for $j \in J_{k_0}$. We now define the isomorphism $T: l_1^n \to E$. Let e_j, $j = 1, \cdots, n$, be a natural basis of l_1 and define

$$Te_j = (uU_k f_j)\left(\frac{\sqrt{\dim E_k}}{\sqrt{n}}\right) \qquad \text{for } j \in J_k.$$

Then $\|T\| = \max_{1 \le j \le n} \|Te_j\| \le K^{(2)}(E) + \alpha\beta\sqrt{\log t_0}$, from above. For scalars $(c_i)_{i=1}^n$, we have

$$\left\|\sum c_j Te_j\right\| \ge \left\|\sum c_j u^{-1} Te_j\right\| = \left(\sum_{i=1}^{k_0}\left\|\sum_{j\in J_i} c_j u^{-1} Te_j\right\|^2\right)^{1/2}$$

$$= \left[\sum_{i=1}^{k_0} \frac{\dim E_i}{n}\left\|\sum_{j \in J_i} c_j \bigcup_i f_j\right\|^2\right]^{1/2}$$

$$= \left[\sum_{i=1}^{k_0} \frac{\dim E_i}{n}\left(\sum_{j\in J_i} |c_j|^2\right)\right]^{1/2} \ge \frac{1}{\sqrt{nk_0}}\sum |c_j|$$

$$\ge \frac{1}{\sqrt{n\tilde{\varphi}(n)}}\sum_{j=1}^{n} |c_j|.$$

That is, $\|T^{-1}\| \le \sqrt{n\tilde{\varphi}(n)}$, completing the proof.

In the results of this section, the spaces that are shown to be relatively close together are, *a priori*, rather far apart. That is, they lie, in a very reasonable sense, on opposite sides of l_2^n. Therefore, in order to find spaces whose Banach–Mazur distance from each other is large, one needs to look at pairs of spaces lying on the same side of l_2^n. We propose candidates for such pairs.

PROBLEM 2. Determine, in each of the cases below, $d(E, l_1^n)$.

(a) Let $\{x_i\}_{i=1}^{2n}$ be distributed on the sphere in Euclidean n-space to maximize the volume of $\mathrm{co}\{\pm x_i\} = B$. Let E be n-space with the norm determined by the unit ball, B.

(b) In l_1^{2n}, let $K_1, \tilde{K}_1$ be a Kashin splitting [14], [25]. That is, K_1 and $\tilde{K}_1$ are nearly Euclidean subspaces of l_1^{2n}, and are orthogonal in the l_2^{2n} sense. Let $E = l_1^{2n}/\tilde{K}_1$. This is, as in (a), an n-dimensional space whose ball has $4n$ extreme points. In this case, the extreme points are, more or less, randomly distributed.

It is known, at least in the second case, that $d(E, l_1^n)$ is asymptotically larger than $\sqrt{n}$ ([8], remark and corollary 2.3 yield the dual form of this statement). There is some evidence that this distance is actually larger than $n^{(1/2)+\varepsilon}$ asymptotically for some $\varepsilon > 0$. It also follows from results of the third-named author that it is hopeless to try to find that spaces are far from l_∞^n by using operator ideal norms. Specifically, if $\gamma_\infty^{(n)}(u)$ is the "ideal norm" generated by factoring rank n operators through l_∞^n, then for any space E, $\gamma_\infty^{(n)}(i_E) \leqq \sqrt{n}$, where i_E denotes the identity on E. (If $u: E \to F$ has rank $\leqq n$, let $u = \Sigma u_i$ with $\mu_i = \beta_i \alpha_i$, $\alpha_i : E \to l_\infty^n$, $\beta_i = l_\infty^n \to F$. Then $\gamma_\infty^{(n)}(u) = \inf\{\Sigma \|\alpha_i\| \|\beta_i\| \mid u = \Sigma u_i\}$.) This statement is precisely dual to factoring the identity through l_1^n.

§2. Factorization through l_∞^n

The results of §1 admit certain extensions. For example, Corollary 1 there tells us that, if E has cotype 2, then the identity map on E factors as $u: E \to l_\infty^n$ followed by $v: l_\infty^n \to E$ so that $\|u\| \|v\| \leqq C\sqrt{n \log n}$. In this section, we see that for any space E with $\dim E = n$, there is a subspace, E_0 of E, whose identity map factors well through l_∞^n, and such that $\dim E_0 \geqq n/10$.

In proving this, we shall use the famous lemma of Dvoretzky and Rogers [4]. We first give a very simple proof of this lemma which is familiar to many experts in the field.

Dvoretzky–Rogers Lemma. *Let E be an n-dimensional space, and let $u: l_2^n \to E$ be defined by $u(B_{l_2^n})$ is the ellipsoid of maximal volume in B_E. Then there is an orthonormal basis, (e_i), in l_2^n so that $\|ue_i\| \geqq (1-(i-1)/n)^{1/2}$.*

Proof. We need only two facts already noted here: $\pi_2(u) = \pi_2(u^{-1}) = \sqrt{n}$ and for any $w: l_2^k \to E$, $\pi_2(w) \leqq \sqrt{k}\|w\|$. Choose $e_1 \in l_2^n$ so that $\|e_1\| = \|ue_1\| = 1$. We proceed recursively. Suppose we have chosen $e_1, e_2, \cdots, e_j$ in l_2^n to be orthonormal and to satisfy the lower bound above. Let $E_j = [e_1, \cdots, e_j]$. Then $u^{-1} \circ (u|_{E_j^\perp})$ is the identity on $E_j^\perp$ so we have $n - j = \operatorname{tr}(u^{-1} \circ (u|_{E_j^\perp})) \leqq \pi_2(u^{-1})\pi_2(u|_{E_j^\perp})$. Since $\pi_2(u^{-1}) = \sqrt{n}$ and $\|u|_{E_j^\perp}\|\sqrt{n-j} \geqq \pi_2(u|_{E_j^\perp}) \geqq (n-j)/\sqrt{n}$, we have $\|u|_{E_j^\perp}\| \geqq \sqrt{n-j}/\sqrt{n}$. We can now select e_{j+1} in $E_j^\perp$ so that $\|e_{j+1}\| = 1$ and $\|ue_{j+1}\| \geqq \sqrt{(n-j)/n}$ as desired.

Theorem 3. *Let E be an n-dimensional space. Then $E \supset E_0$ with $\dim E_0 \geqq n/10$ so that $i_{E_0} = v \circ u$ where $u: E_0 \xrightarrow{u} l_\infty^n \xrightarrow{v} E_0$ and $\|u\| \|v\| \leqq 3\delta\sqrt{n}$, with $\delta = (1 - 3/\sqrt{10})^{-1}$.*

Proof. Let $l_2^n \xrightarrow{u} E \xrightarrow{u^{-1}} l_2^n$ be the maximal ellipsoid factorization as above, and let (e_i) be the orthonormal basis from the Dvoretzky–Rogers lemma. We now define an operator $w: E \to l_2^n$. Let x_j^* in E^* be chosen so that $\|x_j^*\| = 1$ and $x_j^*(ue_j) = \|ue_j\|$. Let

$$w = \left(\frac{n}{\sum \|ue_j\|^2}\right)\left(\sum \|ue_j\| x_j^* \otimes e_j\right).$$

Since $u = \Sigma e_i \otimes (ue_i)$, we see easily that $\operatorname{tr} w \circ u = n$. We now factor w through l_∞^n: Define $E \xrightarrow{w_1} l_\infty^n \xrightarrow{w_2} l_2^n$ with $w = w_2 \circ w_1$, where $w_1(x) = (x_j^*(x))$ and $w_2((\alpha_i)_{i=1}^n) = (n/\Sigma \|ue_j\|^2)(\alpha_i \|ue_i\|)_{i=1}^n$. Using the estimates $\sqrt{(n-j)/n} \leqq \|ue_j\| \leqq 1$, we see that

$$\|w_1\| \leqq 1 \quad \text{and} \quad \|w_2\| = \pi_2(w_2) \leqq \left(\frac{2n}{n-1}\right)\sqrt{n}.$$

Now represent $w \circ u$ in polar form, $\Sigma_{i=1}^n \lambda_i f_i \otimes g_i$, with $\lambda_i \geqq 0$, (f_i) and (g_i) orthonormal. Let $U = \Sigma g_i \otimes f_i$, so U is an isometry on l_2^n, and $U \circ w \circ u = \Sigma \lambda_i f_i \otimes f_i$. We have $n = \operatorname{tr} w \circ u \leqq i_1(w \circ u) = i_1(U \circ w \circ u) = \Sigma \lambda_i$, and $(\Sigma \lambda_i^2)^{1/2} = HS(U \circ w \circ u) = \pi_2(U \circ w \circ u) \leqq \|U\| \pi_2(w_2) \|w_1\| \|u\| \leqq 3\sqrt{n}$ (at least for $n \geqq 3$). That is, considering $\lambda = (\lambda_i)$ to be a function in $L_p(\{1, \cdots, n\}, \mu)$, where $\mu(\{i\}) = 1/n$ for each n, we have $1 \leqq \int \lambda d\mu$ and $(\int \lambda^2 d\mu)^{1/2} \leqq 3$. Let $A_\varepsilon \subset \{1, \cdots, n\}$ be the set where $\lambda_i \geqq \varepsilon$. Then

$$1 = \int_{A_\varepsilon} \lambda d\mu + \int_{A_\varepsilon^c} \lambda d\mu \leqq \mu(A_\varepsilon)^{1/2} \left(\int \lambda^2 d\mu\right)^{1/2} + \varepsilon\mu(A_\varepsilon^c) \leqq \mu(A_\varepsilon)^{1/2} \cdot 3 + \varepsilon.$$

Thus, $\mu(A_\delta) \geqq 1/10$ if $1/\delta = 1 - 3/\sqrt{10}$. That is, the cardinality of the set $A = [\lambda_i \geqq 1/\delta]$ is greater than $n/10$. In l_2^n, let $F = [f_k \mid k \in A]$, so $\dim F \geqq n/10$. Also, for any $f \in F$, we have $\|U \circ w \circ u(f)\| \geqq (1/\delta)\|f\|$. The subspace E_0 is just $u(F)$. Let $v : E_0 \to l_\infty^n$ by $v = w_1|_{E_0}$ and $\tilde{v} : l_\infty^n \to E_0$ by $\tilde{v} = u \circ (U \circ w \circ u|_F)^{-1} \circ (U \circ w_1)$. This is the desired operator.

We conclude the paper with a few remarks. The lemmas of §1 are, in fact, estimates of the Levy mean of the norm of a space over the maximal and minimal volume ellipsoids. In [6] we see that, if $|\cdot|$ is a Euclidean norm on E such that $a|x| \leqq \|x\| \leqq b|x|$ with $a^{-1}b \leqq n$, then the Levy mean, M_r, of the norm of E over the Euclidean sphere $S = \{|x| = 1\}$ is, up to a constant, the same as $(\int_S \|x\|^2 dm)^{1/2}$. For a normalized Gaussian measure on this Euclidean space, we have $(\int_S \|x\|^2 dm)^{1/2} = n^{-1/2}(\int \|x\|^2 d\gamma)^{1/2}$. The right-hand side is just $l(u)$, where $u : (E, |\cdot|) \to (E, \|\cdot\|)$ is the formal identity map.

Following [16] and [7], one can define the l-ellipsoid to be that one given by the relation $l(u)l^*(u^{-1}) = n$. (This can be further normalized by $\|u\| = 1$, if desired.) By [7], [19] and [22], $l^*(u^{-1}) \leqq (\log n) l(u^{*-1})$. By the remarks above, for this ellipsoid one has $M_r M_{r^*} \leqq C \log n$. The l-ellipsoid remains until now a rather mysterious object. For certain computations, it would be very nice to have estimates for M_r and M_{r^*} when $u(B_{l_2^n})$ is, for example, the maximal volume ellipsoid. Our Lemmas 1 and 2 give such estimates. These are summarized in

COROLLARY 2. *Let $u : l_2^n \to E$ so that $u(B_{l_2^n})$ is the ellipsoid of maximal volume in B_E. Then,* (a) $M_r \leqq cK^{(2)}(E)$, (b) $M_{r^*} \leqq cK^{(2)}(E^*)$, *and* (c) $M_{r^*} \leqq cK_{(2)}(E)\log(\dim E)$. *In particular,* $M_rM_{r^*} \leqq cK^{(2)}(E)K_{(2)}(E)\log(\dim E)$.

Part (a) actually just uses the trivial estimate $l(u) \leqq K^{(2)}(E)\pi_2(u)$ noted before Lemma 1.

In case E has enough symmetries, the l-ellipsoid and maximal ellipsoids coincide. Therefore, in that case, the appearances of $K^{(2)}(E)$ and $K_{(2)}(E)$ are superfluous. We do not know whether or not this is true in general.

Added in proof. The authors have recently learned about the striking result of E. D. Gluskin. There is a constant $c > 0$ so that the diameter of the space of all n-dimensional spaces is greater than cn (Funkcional. Anal. i Priložen (Russian), to appear in 1981).

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